The reduction types of absolutely simple abelian threefolds with complex multiplication

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Let E be an elliptic curve defined over a number field k. We say that E has *complex multiplication (CM)* if the endomorphism ring $\mathrm{End}_{\bar{k}}(E)$ of E over $\overline{k}$ is strictly larger than $\mathbb{Z}$. In that case, $\mathrm{End}_{\bar{k}}(E)$ is isomorphic to an order $\mathcal{O}$ in an imaginary quadratic field K.

If E has complex multiplication, then over a finite field extension k' of k, the elliptic curve E/k' has good reduction at every prime of $\mathcal{O}_{k'}$; we say that E has *potential good reduction* everywhere (at every prime). Suppose that E has complex multiplication by the ring of integers $\mathcal{O}_K$ of K. Let $\mathfrak{p} \subset \mathcal{O}_{k'}$ be a prime ideal lying above a rational prime p. Then the reduction $E_{\mathfrak{p}} := E \bmod \mathfrak{p}$ is

- *ordinary* if and only if p completely splits in K;
- *supersingular* if and only if p is ramified or inert in K.

In this project we will be interested in a similar classification for absolutely simple abelian threefolds with complex multiplication. In the case of abelian surfaces, the classification is given by Goren-Lauter [2].

An absolutely simple abelian variety A/k of dimension g is said to have complex multiplication if the endomorphism ring $\mathrm{End}_{\overline{k}}(A)$ is an order in a *CM field* K of degree $2g$. As in the elliptic curve case, an abelian variety A with complex multiplication has *potential good reduction* everywhere, meaning that A has good reduction at every prime over a finite extension k' of the base field k.

Suppose that k' is a finite extension of k over which an abelian variety A with CM has good reduction everywhere. If A is absolutely simple and has CM by the maximal order of a CM field K then the reduction type of A at a prime $\mathfrak{p} \subset \mathcal{O}_{k'}$ is closely related to the decomposition type of $p := \mathfrak{p} \cap \mathbb{Z}$ in $K/\mathbb{Q}$. Studying the decomposition types of primes in sextic CM fields will be the first step towards the classification. When $K/\mathbb{Q}$ is a cyclic extension, the classification is given in [3, Proposition 4.1] up to isogeny and the detailed proof of this proposition is given in the master thesis of Dogger [1, Proposition 4.23].

For each prime decomposition type, we will determine the endomorphism algebra of the reduction $A_{\mathfrak{p}}$ of A at primes $\mathfrak{p}$ of this type, and its corresponding p-rank and a-number. We will also look at some explicit examples, such as Jacobians of genus-3 curves.

Remarks for the participants:

- If you are not familiar with number fields, you should at least study Chapters 1,2,3 and 8 of Stevenhagen [7] or related chapters of any number theory book.
- If you are not familiar with abelian varieties, watching the videos of Anna Somoza would be a good start: https://www.carmin.tv/fr/collections/anna-somoza-complex-abelian-varieties
- If you are not familiar with CM theory, you may read the first chapter of the PhD thesis of Streng [8]. For a detailed study, you may study Milne [5], Lang [4], Shimura-Taniyama [6].
- A classification of p-ranks of abelian threefolds can be found in the master thesis of Dogger [1].
- A more detailed project description will be provided to the participants.

References

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